An Empirical Model of Ionospheric Scintillation at High Latitudes

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   - Chaoticity model
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The ionosphere

- The ionized part of the upper atmosphere extends from \( \approx 60 \) – 1000 km
- Photo-ionization by the sun
- Diurnal variations
- Dependence on the solar activity
- Different layers characterized by density maxima at certain altitudes
Scintillation

\[ \Delta \vec{E} + k^2 [1 + \epsilon_1(\vec{r})] \vec{E} = 0 \]  

1. The non-homogeneous aspect of the electron density irregularities induces scattering (diffraction, refraction)
2. Trans-ionospheric radio waves experience perturbations in the power and phase components
3. Information about the dynamics of the ionospheric plasma can be extracted from the scattered signal
Scintillation are predominant at equatorial and auroral regions

1 J. AERONS (1982)
Why study scintillation?

- Electron density irregularities induce a phase advance and a group delay of the electromagnetic wave

\[
\begin{align*}
n_p &= 1 - \frac{40.3n_e}{f^2} \\
n_g &= 1 + \frac{40.3n_e}{f^2}
\end{align*}
\]

\[
\begin{align*}
\Delta S_{\text{iono}, p} &= \frac{-40.3 \int_{SV}^{\text{User}} n_e dl}{f^2} \\
\Delta S_{\text{iono}, g} &= \frac{40.3 \int_{SV}^{\text{User}} n_e dl}{f^2}
\end{align*}
\]

- Resolve the navigational problem
Global Positioning System (GPS)

- Currently 32 satellites in orbit in 6 orbital planes
- Transmit coded signal in L-band: 1227 and 1575 MHz
The Canadian High Arctic Ionospheric Network (CHAIN)

- 24 stations in the Arctic region
- 6 Canadian Advanced Digital Ionosonde (CADI)
- 10 equipped with NovAtel GSV4004B dual-frequency GPS receiver
The Canadian High Arctic Ionospheric Network (CHAIN)

NovAtel GSV4004B dual-frequency GPS receiver

Sampling rate of 50 Hz
The Canadian High Arctic Ionospheric Network (CHAIN)

Current approaches

Indices quantifying the scintillation

- Amplitude $S_4 = \sqrt{\frac{\langle P^2 \rangle - \langle P \rangle^2}{\langle P \rangle^2}}$
- Phase
  $$\sigma_{\phi} = \sqrt{\langle \phi^2 \rangle - \langle \phi \rangle^2}$$
- $\sigma_{CHAIN} = \sqrt{\langle (\frac{\delta \phi}{\delta t})^2 \rangle |\phi|}$

Mushini et al. (2013)
Current approaches

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- Phase $\sigma_\phi = \sqrt{\langle \phi^2 \rangle - \langle \phi \rangle^2}$
- $\sigma_{CHAIN} = \sqrt{\langle \left( \frac{\delta \phi}{\delta t} \right)^2 \mid \phi \rangle}$

Mushini et al. (2013)

These indices do not completely quantify the dynamics of the system
The model

**Empirical model**
- Optimize the extraction of the scintillation components
- Quantify the chaoticity of the ionospheric scintillation
- Characterize the seasonal and spatial variation of the scintillation occurrences

**Numerical model**
- Development of a simulator of the trans-ionospheric channel
The model

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An intermittency model

- Intermittency is a central problem in turbulence
- Scintillation exhibit an intermittent aspect
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Objectives:
- Parametrize intermittent scintillation
- Quantify the intermittency
An intermittency model

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- Quantify the intermittency
Turbulence
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Turbulence:

- Power spectra is characterized by 3 regions
- Power law
- Constant energy transfer rate $\epsilon$
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- Power spectra is characterized by 3 regions
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Intermittent turbulence

Castaing’s assumptions:

- Intermittency can be studied by the way of measurement of signal differences: \( F(\tau, t) = x(t + \tau) - x(t) \)

- Energy transfer rate is scale-dependent: the PDF is a superposition of normal distribution whose variances are log-normally distributed

\[
P_\lambda(\delta F(\tau, t)) = \frac{1}{\lambda 2 \pi} \int_0^\infty \exp\left(-\frac{\delta F(\tau, t)^2}{2\sigma^2}\right) \exp\left(-\frac{\ln^2(\sigma / \sigma_0)}{2\lambda^2}\right) \frac{d\sigma}{\sigma^2}
\]
First time such a similarity is observed between the fluid turbulence and ionospheric scintillation
Higher order moments

A simple intermittency model

\[ \delta A(\tau, t) = 0 \text{ (no intermittency)} \]

\[ \delta A(\tau, t) = x_0 \text{ (intemittency)} \]
Higher order moments

A simple intermittency model

\( \delta A(\tau, t) = 0 \) (no intermittency)

\( \delta A(\tau, t) = x_0 \) (intemittance)

\[
P(x; \tau, t) = \frac{d}{dx} \left[ prob\{\delta A(\tau, t) \leq x\} \right]
\]

\[
P(x; \tau, t) = \Gamma(\tau, t)\delta(x - x_0) + (1 - \Gamma(\tau, t))\delta(x)
\]
Higher order moments

A simple intermittency model

\[ \delta A(\tau, t) = 0 \] (no intermittency)
\[ \delta A(\tau, t) = x_0 \] (intemittency)

\[ P(x; \tau, t) = \frac{d}{dx} \left[ \text{prob}\{\delta A(\tau, t) \leq x\} \right] \] (2)

\[ P(x; \tau, t) = \Gamma(\tau, t) \delta(x - x_0) + (1 - \Gamma(\tau, t)) \delta(x) \] (3)

\[ < [\delta A(\tau, t)]^n > = \frac{\mu_n}{\sigma^n} \] (4)

\[ S^2 = \frac{(x_0 - 2\mu)^2}{\mu(x_0 - \mu)} \]

\[ K = \frac{(x_0^2 - 3x_0\mu + 3\mu^2)}{\mu(x_0 - \mu)} = S^2 + 1 \] (5)
Correlation between the skewness and kurtosis

\[ K = 5.5S^2 - 1.4S + 3.6 \]
\[ K = 4 \sinh^2 \left[ \frac{1}{2} \frac{\tau_i}{\tau} \right] + 1 \]
Chaoticity model
Chaoticity model

- Filter out the non-scintillation components (Large scale fluctuations, > 100 km )
- Construct the mathematical tools to quantify the chaoticity of the ionospheric plasma during scintillation
- How does the chaoticity vary?
Chaoticity model

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- How does the chaoticity vary?
Optimization of the cut-off scale

Wavelet transform:
- Excellent tool for analyzing localized variations of power within a time series

\[
w(s, \tau) = \frac{1}{\sqrt{s}} \int_{l} \psi_{s,t}^* x(t) dt \quad (6)
\]

\[
\epsilon(s, \tau) = |w(s, \tau)|^2 \quad (7)
\]

- Turbulent events are singled out clearly during scintillation
Optimization of the cut-off scale

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\[ w(s, \tau) = \frac{1}{\sqrt{s}} \int_{l}^{u} \psi_{s,t}^* x(t) dt \]  \hspace{1cm} (6)

\[ \epsilon(s, \tau) = |w(s, \tau)|^2 \]  \hspace{1cm} (7)

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Wavelet transform:

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\[ w(s, \tau) = \frac{1}{\sqrt{s}} \int_I \psi_{s,t}^* x(t) dt \]  \hspace{1cm} (6)

\[ \epsilon(s, \tau) = |w(s, \tau)|^2 \]  \hspace{1cm} (7)

- Turbulent events are singled out clearly during scintillation
Optimum cut-off scale

The raw signal is composed of different contributions:

- Trend (large scales)
- Scintillation (intermediate scales)
- High frequency noise

- Single out scintillation components from diurnal variation (large scale fluctuations)
- Entropy maximization
Entropy construction

Amplitude: the general Tsallis entropy

\[ S_q = k_B \frac{1 - \sum_{i=1}^{N} p_i^q}{q - 1} \]  

(8)

We sub-divide the system into two sub-systems: scintillation and non-scintillation contributions

\[ S_q(A + B) = S_q(A) + S_q(B) + (1 - q)S_q(A)S_q(B) \]  

(9)

Wavelet-based PDFs:

\[ p_{scin}(s, s_{opt}, s_{min}, t_{max}) = \frac{\sum_{t=0}^{t_{max}} \epsilon(s, t)}{\sum_{t=0}^{t_{max}} \sum_{s=s_{min}}^{s_{opt}} \epsilon(s, t)} \]  

(10)

\[ p_{bg}(s, s_{opt}, s_{max}, t_{max}) = \frac{\sum_{t=0}^{t_{max}} \epsilon(s, t)}{\sum_{t=0}^{t_{max}} \sum_{s=s_{opt}+1}^{s_{max}} \epsilon(s, t)} \]  

(11)
Maximization of the entropy

\[ s_{\text{opt}} = \arg \max (S) \quad (12) \]

A maximum for the total entropy \( S \), for given \( s_{\text{min}} \) and \( s_{\text{max}} \) can be translated into

\[
\frac{\partial S_q}{\partial s_{\text{opt}}} = [1 + (1 - q)S_{q-\text{scin}}] \frac{\partial S_{q-\text{bg}}}{\partial s_{\text{opt}}} + [1 + (1 - q)S_{q-\text{bg}}] \frac{\partial S_{q-\text{scin}}}{\partial s_{\text{opt}}} = 0 \quad (13)
\]

\[
\frac{\partial^2 S_q}{\partial s_{\text{opt}}^2} \leq 0 \quad (\text{Maximum})
\]
The probability density function

Defining $q$

\textbf{Variational principal}

- $\sum_{i=1}^{N} p(s_i) = 1$
- $\sum_{i=1}^{N} p(s_i) s_i = \langle s \rangle$
- $\Gamma = S_q + \alpha \sum_{i=1}^{N} p(s_i) - \alpha \beta (q - 1) \sum_{i=1}^{N} p(s_i) s_i$
- $\delta \Gamma / \delta p_i = 0, \forall i$

$$p(s_i) = \frac{(1 - \beta (q - 1) s_i)^{\frac{1}{q-1}}}{Z_q}$$

(14)
The phase

Gaussian statistics

\[ S = -k_B \sum_{i=1}^{N} p(s_i) \ln p(s_i) \]  

\[ s_{opt} = s_{min} + \frac{1}{\beta} \ln \left( \frac{1 + e^{-\beta}}{1 + e^{-\beta(s_{max} - s_{min})}} \right) \]
Distribution of the optimum scale

- Scintillation events recorded from 2009 to 2011
- Use the 10 CHAIN stations

<table>
<thead>
<tr>
<th>Station</th>
<th>Corrected Geomag. Lat. (°N)</th>
<th>Corrected Geomag. Long. (°E)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eureka</td>
<td>88.10</td>
<td>329.86</td>
</tr>
<tr>
<td>Resolute Bay</td>
<td>83.13</td>
<td>320.61</td>
</tr>
<tr>
<td>Pond Inlet</td>
<td>81.67</td>
<td>0.69</td>
</tr>
<tr>
<td>Cambridge Bay</td>
<td>77.24</td>
<td>310.11</td>
</tr>
<tr>
<td>Taloyoak</td>
<td>78.75</td>
<td>329.70</td>
</tr>
<tr>
<td>Hall Beach</td>
<td>78.33</td>
<td>353.45</td>
</tr>
<tr>
<td>Qikiqtarjuaq</td>
<td>75.77</td>
<td>22.57</td>
</tr>
<tr>
<td>Iqaluit</td>
<td>72.85</td>
<td>14.04</td>
</tr>
<tr>
<td>Sanikiluaq</td>
<td>67.19</td>
<td>356.44</td>
</tr>
</tbody>
</table>
Joint probability density function
- Seasonal variation, with a transition at the equinoxes
- Discrepancies in the mean value of the optimum scale
Scintillation entropy

\[
S_{sc} = k_B \frac{1 - \sum_{s=s_{\text{opt}}}^{s=s_{\text{min}}} p_s^q}{q - 1}
\]

- Construct the entropy for the scintillation components
- Quantify the seasonal variation of the chaoticity of ionospheric scintillation
- Scintillation map construction in the geomagnetic domain
- Entropy is higher during summer
- The phase component is more pronounced
Phase

- Fall
- Winter
- Summer
- Spring
Power

Fall

Winter

Summer

Spring
Another index to quantify the chaoticity of scintillation: the wavelet-based fractal dimension

$$D(T, a : b) = \frac{\int_0^T \int_a^b \epsilon(s, t) dt ds}{\int_0^T \int_{s_{\text{min}}}^{s_{\text{max}}} \epsilon(s, t) dt ds} \tag{18}$$
Variations of the wavelet-based fractal dimension
Power

Fall

Winter

Summer

Spring
Phase

Fall

Winter

Summer

Spring
Conclusion on the distribution of the chaoticity

- Symmetry in the chaoticity of the power component during summer, fall, spring
- Asymmetry of the distribution of the chaoticity of the phase component in summer
- Asymmetry of the chaoticity distribution of the power and phase components during winter
- Predominance of the chaoticity in the auroral region
Wave equation

- Solve the wave equation
- Build a simulator of trans-ionospheric channel

Helmholtz equation:

\[
[\nabla^2 + k^2 (1 + \epsilon_1(\rho_m, z))u(\rho, z) = 0 \tag{19}
\]

\[
\epsilon_1(\rho_m, z) = -\left(\frac{\omega_p}{\omega}\right)^2\frac{[\nabla N(\rho + \tan \theta \hat{a}_\perp z, z)/N_0]}{1 - (\omega_p/\omega)^2} \tag{20}
\]

\[
u(\rho, z) = U(\rho, z)\exp(jkz) \tag{21}
\]
\[ 2jk \frac{\partial U(\vec{\rho}, z)}{\partial z} + \left( \frac{d^2}{dx^2} + \frac{d^2}{dy^2} \right) U(\vec{\rho}, z) + k^2 \epsilon_1(\vec{\rho}_m, z) U(\vec{\rho}, z) = 0 \]
The split step method

Inside the irregularities:

\[ 2jk \frac{\partial U(\tilde{\rho}, z)}{\partial z} + k^2 \epsilon_1(\tilde{\rho}_m, z) U(\tilde{\rho}, z) = 0 \quad (23) \]

\[ U(\tilde{r}_\perp, z_1) = U(\tilde{\rho}, z_0) \exp(j \frac{k}{2} \int_{z_0}^{z_1} \sec \theta \epsilon_1(\tilde{\rho} + \tan \theta \hat{a}_\perp z, z) dz) \quad (24) \]

Free propagation:

\[ 2jk \frac{\partial U(\tilde{\rho}, z)}{\partial z} + \left( \frac{d^2}{dx^2} + \frac{d^2}{dy^2} \right) U(\tilde{\rho}, z) = 0 \quad (25) \]
Hybrid spectrum\(^3\):

\[
P(k') = \frac{a(p_H^{-2})}{2\pi^{3/2}k_0^2} \Delta N(\bar{\rho} + \tan \theta \hat{a}_z, z)^2(1 + \frac{k_x'^2 + k_y'^2}{k_0^2})^{-p_H/2} \exp\left(-\left(ak\right)^2 \frac{k_z'^2}{k_0^2}\right)
\]

**Characteristics:**

- Power law characterized by spectral index \(p_H\)

**Geometry:**

- Rod-shaped irregularities
- Axial ratio is given by \(ak\)

\(^3\)Costa (1977)
Time series

Assuming a "frozen-in" aspect of the field, we can express that motion by translating the coordinates of the plasma bulk:

\[ \vec{r} \rightarrow \vec{r} + \vec{v}_d t \]  

(26)
Breakdown of the power law for the amplitude component of the signal at the Fresnel frequency
Conclusion and future work

Intermittency model
- Castaing function fits intermittent events
- Amplitude scintillation are more intermittent at smaller scales

Entropy model
Amplitude and phase present different statistics:
- The amplitude component is characterized by the general Tsallis entropy
- The phase component is described by the Boltzmann-Gibbs entropy

Seasonal variation:
- Both phase and amplitude exhibit seasonal variations
- Discrepancies in the value of the entropy for the phase and amplitude components
Conclusion and future work

**Optimum scale**

- Discrepancies in the values of the optimum scale for the phase and the power components
- The distribution of the optimum scale exhibits two peaks while the power exhibit one

**Simulator of the trans-ionospheric channel**

- Takes into account the peculiar geometry at high latitudes
- Use the split step method with and explicit stencil (not computationally expensive)
- Reproduces the spectral characteristics of the scintillating signal at high latitudes
Future work

- Interhemispheric comparison
- Extend the analysis for high solar activity periods
- Solve the inverse problem
Electron density highly inhomogeneous

- Particles precipitations
- Instabilities

\[ \epsilon(\vec{r}, t) = \epsilon_0(\vec{r}) + \epsilon(\vec{r}, t) \]  

(27)
Maxwell’s equations

\[ \nabla \times \nabla \times \vec{E} = -\frac{1}{c^2} \frac{\partial^2(\epsilon \vec{E})}{\partial t^2} - \frac{4\pi}{c} \frac{\partial \vec{J}}{\partial t} \]  \tag{28} \\
\[ \nabla \times \nabla \times \vec{E} = -\nabla^2 \vec{E} + \nabla(\nabla \cdot \vec{E}) \]  \tag{29} \\
\[ \nabla \cdot \vec{D} = \nabla \cdot (\epsilon \vec{E}) = \vec{E} \cdot \nabla \epsilon + \epsilon \nabla \cdot \vec{E} = 0 \]  \tag{30} \\
\[ \nabla \cdot \vec{E} = -\vec{E} \cdot \nabla (\log \epsilon) \]  \tag{31} \\
\[ \nabla^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} [(1 + \delta \epsilon) \vec{E}] = \frac{4\pi}{c} \frac{\vec{J}}{\partial t} - \nabla \vec{E} \cdot \nabla [\log(1 + \delta \epsilon)] \]  \tag{32} \\

We drop the depolarization term
\[ \nabla^2 \tilde{E}(\vec{r}) - E(\vec{r}) \frac{1}{c^2} \exp(i\omega t) \frac{\partial^2}{\partial t^2} [1 + \Delta \varepsilon(\vec{r}, t)] \exp(-i\omega t) = -\frac{4\pi i\omega}{c} J(\vec{r}) \quad (33) \]

\[ \frac{1}{c^2} \exp(i\omega t) \frac{\partial^2}{\partial t^2} [1 + \Delta \varepsilon(\vec{r}, t)] \exp(-i\omega t) = \quad (34) \]

\[ k^2 [1 + \Delta \varepsilon(\vec{r}, t)] - \frac{2ik}{c} \frac{\partial}{\partial t} \Delta \varepsilon(\vec{r}, t) + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \Delta \varepsilon(\vec{r}) \quad (35) \]
Taylor approximation

\[
\frac{\partial}{\partial t} \Delta \epsilon(\vec{r}, t) \approx \left( \frac{v}{l} \right) \Delta \epsilon
\]

+ Doppler shift

\[
\frac{\partial}{\partial t} \Delta \epsilon(\vec{r}, t) \approx v \Delta \epsilon \left( \frac{1}{l} + \frac{1}{\lambda} \right)
\]

\[k^2 \Delta \epsilon \gg \Delta \epsilon \frac{2vk}{c} \left( \frac{1}{l} + \frac{1}{\lambda} \right)\]
\[ \Delta \vec{E} + k^2[1 + \epsilon_1(\vec{r})]\vec{E} = 0 \] (38)

Complex amplitude
\[ \vec{E} = \vec{u}(\vec{r})\exp(-ikz) \] (39)

\[ -2jk \frac{\partial u}{\partial z} + \nabla^2 u = -k^2\epsilon_1(\vec{r})u \] (40)

Parabolic approximation

0 < z < L
\[ -2jk \frac{\partial u}{\partial z} + \nabla_{\perp}^2 u = -k^2\epsilon_1(\vec{r})u \] (41)

z > L
\[ -2jk \frac{\partial u}{\partial z} + \nabla_{\perp}^2 u = 0 \] (42)
Phase screen theory

The ionosphere acts as a phase changing screen

\[ u_0(\vec{\rho}) = A_0 \exp[-j\phi(\vec{\rho})] \]  \hspace{1cm} (43)

\[ \phi(\vec{\rho}) = k_0(\Delta \phi) = -\lambda r_e \Delta N_T(\vec{\rho}) \]  \hspace{1cm} (44)

Fresnel diffraction

\[ u(\vec{\rho}, z) = \frac{jkA_0}{2\pi z} \int \int \exp\left[-j\phi(\vec{\rho}') + \left(\frac{k}{2z}\right) |\vec{\rho} - \vec{\rho}'| \right] d^2\rho' \]  \hspace{1cm} (45)
The General Tsallis entropy

\[ S_q = k_B \frac{1 - \sum_{i=1}^{N} p_i^q}{q - 1} \]  (46)

- \( p_i \): the probability associated with the microscopic configurations
- \( N \): the total number of elements
- \( q \): a real number, quantifying the non-extensivity
- \( k_B \): the Boltzmann constant
The system is hence sub-divided into two subsystem, namely, the scintillation and non-scintillation system, A and B, respectively. The general Tsallis entropy can be, then, given by

\[ S_q(A + B) = S_q(A) + S_q(B) + (1 - q)S_q(A)S_q(B) \]  

\[(47)\]
In order to define criteria for the optimum scale, we need to define the expressions of the probabilities associated with scintillation and background variability respectively. Let’s denote by $s_{opt}$ the optimum cut-off scale delimiting the scintillation from the trend. Then the probabilities for scintillation and background are given, respectively, by:

$$p_{scin}(s, s_{opt}, s_{min}, t_{max}) = \frac{\sum_{t=0}^{t=t_{max}} \sum_{s=s_{min}}^{s=s_{opt}} \epsilon(s, t)}{\sum_{t=0}^{t=t_{max}} \sum_{s=s_{min}}^{s=s_{opt}} \epsilon(s, t)}$$ (48)

$$p_{bg}(s, s_{opt}, s_{max}, t_{max}) = \frac{\sum_{t=0}^{t=t_{max}} \sum_{s=s_{opt}+1}^{s=s_{max}} \epsilon(s, t)}{\sum_{t=0}^{t=t_{max}} \sum_{s=s_{opt}+1}^{s=s_{max}} \epsilon(s, t)}$$ (49)
The total entropy is then given by the sum of the two entropies associated to both components and a term representing the non-extensivity, proportional to the product of the two:

\[ S_q(s_{\text{min}}, s_{\text{max}}, s_{\text{opt}}) = S_{q-\text{scin}} + S_{q-\text{bg}} + (1 - q)S_{q-\text{scin}}S_{q-\text{bg}} \] (50)
The optimum scale criterion is the maximization of the information for both systems, hence the scale for which the total entropy is a maximum:

$$s_{opt} = \text{argmax}(S)$$  \hspace{1cm} (51)

A maximum for the total entropy $S$, for given $s_{min}$ and $s_{max}$ can be translated into

$$\frac{\partial S_q}{\partial s_{opt}} = [1 + (1 - q) S_{q-scin}] \frac{\partial S_{q-bg}}{\partial s_{opt}} + [1 + (1 - q) S_{q-bg}] \frac{\partial S_{q-scin}}{\partial s_{opt}} = 0 \hspace{1cm} (52)$$

and

$$\frac{\partial^2 S_q}{\partial s_{opt}^2} \leq 0 \hspace{1cm} (53)$$
which in turn lead to

\[
\left[ \frac{p_{bg}(s_{opt} + 1)}{p_{scin}(s_{opt})} \right]^q = \frac{1 + (1 - q)S_{q-bg}(s_{opt} + 1)}{1 + (1 - q)S_{q-scin}(s_{opt})} = \frac{\sum_{s_{opt}+1}^{s_{max}} p_{bg}(s)^q}{\sum_{s_{min}}^{s_{opt}} p_{scin}(s)^q}
\] (54)

This clearly defines an equation for the optimum scale \( s_{opt} \). Moreover, using equation 52, one can show that equation 53 reduces to

\[
\frac{\partial^2 S_q}{\partial s_{opt}^2} = -2k_B \frac{p_{scin}(s_{opt})^q p_{bg}(s_{opt} + 1)^q}{1 - q}
\] (55)

which remains negative for all \( q < 1 \), and assures a maximum for the entropy at \( s_{opt} \).
We follow the approach used by [Tsallis, 1988] in the determination of the entropy. In order to optimize $S_q$ we impose the conditions:

$$
\sum_{i=1}^{N} p(s_i) = 1
$$

(56)

where $N$ is the number of scale in the system, $s_1 = s_{\text{min}}$, $s_N = s_{\text{max}}$, and we have set $k_B = 1$.

$$
\sum_{i=1}^{N} p(s_i)s_i = \langle s \rangle
$$

(57)

And define the quantity $\Gamma$ as follows:

$$
\Gamma = S_q + \alpha \sum_{i=1}^{N} p(s_i) - \alpha \beta (q - 1) \sum_{i=1}^{N} p(s_i)s_i
$$

(58)
Conclusion and future work

Using a variational principle imposing the condition $\delta \Gamma / \delta p_i = 0$, $\forall i$, one can determine the expression for the probability:

$$p(s_i) = \frac{(1 - \beta(q - 1)s_i)^{\frac{1}{q-1}}}{Z_q}$$

(59)

With the partition functions given by:

$$Z_q = \sum_{i=1}^{N}[1 - \beta(q - 1)s_i]^{\frac{1}{q-1}}$$

$$Z_{q-scint} = \sum_{s_{min}}^{s_{opt}}[1 - \beta(q - 1)s_i]^{\frac{1}{q-1}}$$

$$Z_{q-bg} = \sum_{s_{opt}+1}^{s_{max}}[1 - \beta(q - 1)s_i]^{\frac{1}{q-1}}$$

(60)
Unlike the power, the phase component of the GPS L1 signal presents a quasi-Gaussian behavior. In order to illustrate this fact, in Figure 2, we construct the probability density function for the time series of the recorded GPS L1 signal, during the scintillation event at Qikiqtarjuaq.

Figure: The fit of $p(s_i)$ for the phase, solid line, $\beta = \frac{1}{\sigma^2} = 0.37$. The distribution presents a quasi-Gaussian statistics, with a kurtosis, $k=3.03$, and a skewness, $s=0.08$. 
Therefore, the entropy functional that best characterizes the phase is the Boltzmann-Gibbs entropy, defined as follows:

\[
S = -k_B \sum_{i=1}^{N} p(s_i) \ln p(s_i)
\]  

(61)

The system is then subdivided into two subsystems, as in the previous section. For the same event, the entropy variation is examined and analyzed. A similar analysis, as for the power, can easily be reproduced for the case when the classical Boltzmann-Gibbs formalism is applicable, such as in the case of the phase. The result for the optimum scale is given by the following expression

\[
1 = \frac{\sum_{s_{\text{min}}}^{s_{\text{max}}} e^{-\beta(s_i-1)} e^{-\beta s_{\text{opt}}}}{\sum_{s_{\text{min}}}^{s_{\text{opt}}} e^{-\beta s_i}}
\]  

(62)

or

\[
\sum_{s_{\text{min}}}^{s_{\text{opt}}} e^{-\beta s_i} = \sum_{s_{\text{opt}}+1}^{s_{\text{max}}} e^{-\beta(s_i-1)}
\]  

(63)

By making the following change of variable

\[
x_i = s_i - s_{\text{min}}
\]

\[
y_i = s_i - s_{\text{opt}} - 1
\]
Conclusion and future work

After equating the two terms, and some algebra, one can recover the following expression for the optimum scale:

\[
S_{opt} = S_{min} + \frac{1}{\beta} \ln \left( \frac{1 + e^{-\beta}}{1 + e^{-\beta}(s_{max} - s_{min})} \right)
\]
Let us define the fractal dimension of the signal for a given time interval $T$ as follows:

$$D(T, a : b) = \frac{\int_0^T \int_a^b \epsilon(s, t) dtds}{\int_0^T \int_{s_{\min}}^{s_{\max}} \epsilon(s, t) dtds}$$  \hspace{1cm} (69)
In order to parametrize the operation of shifting and scaling, the wavelet is written in the following form:

$$\psi_{a, b}(t) = \frac{1}{\sqrt{a}} \psi\left(\frac{t - b}{a}\right)$$

with ”a” being the scaling parameter and ”b” representing the shifting parameter. The wavelet is then convolved with the signal to obtain the wavelet transform:

$$\gamma(a, b) = \int_{-\infty}^{+\infty} f(t) \frac{1}{\sqrt{a}} \psi^*(\frac{t - b}{a}),$$